

A GENERALIZATION OF THE WEAK AMENABILITY OF SOME BANACH ALGEBRA

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ABSTRACT. Let A be a Banach algebra and A^{**} be the second dual of it. We show that by some new conditions, A is weakly amenable whenever A^{**} is weakly amenable. We will study this problem under generalization, that is, if $(n+2)$ -th dual of A , $A^{(n+2)}$, is $T-S$ -weakly amenable, then $A^{(n)}$ is $T-S$ -weakly amenable where T and S are continuous linear mappings from $A^{(n)}$ into $A^{(n)}$.

1. Preliminaries and Introduction

Let A be a Banach algebra and A^* , A^{**} , respectively, are the first and second dual of A . For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $\langle a'a, b \rangle = \langle a', ab \rangle = a'(ab)$ and $\langle aa', b \rangle = \langle a', ba \rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a' \rangle = \langle a', a \rangle$ for every $a \in A$ and $a' \in A^*$. Arens [1] has shown that given any Banach algebra A , there exist two algebra multiplications on the second dual of A which extend multiplication on A . In the following, we introduce both multiplication which are given in [13]. The first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\begin{aligned}\langle a'a, b \rangle &= \langle a', ab \rangle, \\ \langle a''a', a \rangle &= \langle a'', a'a \rangle, \\ \langle a''b'', a' \rangle &= \langle a'', b''a' \rangle.\end{aligned}$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\begin{aligned}\langle aoa', b \rangle &= \langle a', ba \rangle, \\ \langle a'oa'', a \rangle &= \langle a'', aoa' \rangle, \\ \langle a''ob'', a' \rangle &= \langle b'', a'ob'' \rangle.\end{aligned}$$

for all $a, b \in A$ and $a' \in A^*$.

We say that A is Arens regular if both multiplications are equal. Let a'' and b'' be elements of A^{**} . By *Goldstine's* Theorem [6, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = \text{weak}^* - \lim_\alpha a_\alpha$ and $b'' = \text{weak}^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', a_\alpha b_\beta \rangle = \langle a''b'', a' \rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', a_\alpha b_\beta \rangle = \langle a''ob'', a' \rangle.$$

2000 *Mathematics Subject Classification.* 46L06; 46L07; 46L10; 47L25.

Key words and phrases. Arens regularity, weak topological centers, weak amenability, derivation.

Thus A is Arens regular if and only if for every $a' \in A^*$, we have

$$\lim_{\alpha} \lim_{\beta} \langle a', a_{\alpha} b_{\beta} \rangle = \lim_{\beta} \lim_{\alpha} \langle a', a_{\alpha} b_{\beta} \rangle .$$

For more detail see [6, 13, 15].

Let X be a Banach A – *bimodule*. A derivation from A into X is a bounded linear mapping $D : A \rightarrow X$ such that

$$D(xy) = xD(y) + D(x)y \text{ for all } x, y \in A.$$

The space of continuous derivations from A into X is denoted by $Z^1(A, X)$.

Easy example of derivations are the inner derivations, which are given for each $x \in X$ by

$$\delta_x(a) = ax - xa \text{ for all } a \in A.$$

The space of inner derivations from A into X is denoted by $N^1(A, X)$. The Banach algebra A is said to be a *amenable*, when for every Banach A – *bimodule* X , the inner derivations are only derivations existing from A into X^* . It is clear that A is amenable if and only if $H^1(A, X^*) = Z^1(A, X^*)/N^1(A, X^*) = \{0\}$.

A Banach algebra A is said to be a *weakly amenable*, if every derivation from A into A^* is inner. Similarly, A is weakly amenable if and only if $H^1(A, A^*) = Z^1(A, A^*)/N^1(A, A^*) = \{0\}$.

Suppose that A is a Banach algebra and X is a Banach A – *bimodule*. According to [5, pp.27 and 28], X^{**} is a Banach A^{**} – *bimodule*, where A^{**} is equipped with the first Arens product.

Let $A^{(n)}$ and $X^{(n)}$ be n – *th dual* of A and X , respectively. By [19, page 4132-4134], if $n \geq 0$ is an even number, then $X^{(n)}$ is a Banach $A^{(n)}$ – *bimodule*. Then for $n \geq 2$, we define $X^{(n)}X^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for all $x^{(n)} \in X^{(n)}$, $x^{(n-1)} \in X^{(n-1)}$ and $a^{(n-2)} \in A^{(n-2)}$ we define

$$\langle x^{(n)}x^{(n-1)}, a^{(n-2)} \rangle = \langle x^{(n)}, x^{(n-1)}a^{(n-2)} \rangle .$$

If n is odd number, then for $n \geq 1$, we define $X^{(n)}X^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $x^{(n)} \in X^{(n)}$, $x^{(n-1)} \in X^{(n-1)}$ and $a^{(n-1)} \in A^{(n-1)}$ we define

$$\langle x^{(n)}x^{(n-1)}, a^{(n-1)} \rangle = \langle x^{(n)}, x^{(n-1)}a^{(n-1)} \rangle .$$

If $n = 0$, we take $A^{(0)} = A$ and $X^{(0)} = X$.

Now let X be a Banach A – *bimodule* and $D : A \rightarrow X$ be a derivation. A problem which is of interest is under what conditions D'' is again a derivation. In [14, 5.9], this problem has been studied for the spacial case $X = A$, and they showed that D'' is a derivation if and only if $D''(A^{**})A^{**} \subseteq A^*$. We study this problem in the generality, that is, if $A^{(n+2)}$ is $T - S$ –weakly amenable, then it follows that $A^{(n)}$ is $T - S$ – weakly amenable where T and S are continuous linear mapping from $A^{(n)}$ into $A^{(n)}$ and $n \geq 0$.

The main results of this paper can be summarized as follows:

- a) Assume that A is a Banach algebra and $A^{(n+2)}$ has $T - w^*w$ property. If $A^{(n+2)}$ is weakly $T'' - S''$ –amenable, then $A^{(n)}$ is weakly $T - S$ –amenable.
- b) Let X be a Banach A – *bimodule* and let $T, S : A^{(n)} \rightarrow A^{(n)}$ be continuous linear mappings. Let the mapping $a^{(n+2)} \rightarrow x^{(n+2)}T''(a^{(n+2)})$ be *weak*-to-weak* continuous for all $x^{(n+2)} \in X^{(n+2)}$. Then if $D : A^{(n)} \rightarrow X^{(n+1)}$ is a $T - S$ – *derivation*, it follows that $D'' : A^{(n+2)} \rightarrow X^{(n+3)}$ is a $T'' - S''$ – *derivation*.

- c) Let X be a Banach A -bimodule and the mapping $a'' \rightarrow x''a''$ be *weak*-to-weak* continuous for all $x'' \in X^{**}$. If $D : A \rightarrow X^*$ is a derivation, then $D''(A^{**})X^{**} \subseteq A^*$.
- d) Let X be a Banach A -bimodule and $D : A \rightarrow X^*$ be a derivation. Suppose that $D'' : A^{**} \rightarrow X^{***}$ is surjective derivation. Then the mapping $a'' \rightarrow x''a''$ is *weak*-to-weak* continuous for all $x'' \in X^{**}$.
- e) Suppose that X is a Banach A -bimodule and A is Arens regular. Assume that $D : A \rightarrow X^*$ is a derivation and surjective. Then $D'' : A^{**} \rightarrow X^{***}$ is a derivation if and only if the mapping $a'' \rightarrow x''a''$ is *weak*-to-weak* continuous for all $x'' \in X^{**}$. In every parts of this paper, $n \geq 0$ is even number.

2. Weak amenability of Banach algebras

Definition 2-1. Let X be a Banach A -bimodule and T, S be continuous linear mappings from A into itself. We say that $D : A \rightarrow X$ is $T-S$ -derivation, if

$$D(xy) = T(x)D(y) + D(x)S(y) \text{ for all } x, y \in A.$$

Now let $x \in A$. Then we say that the linear mapping $\delta_x : A \rightarrow A$ is inner $T-S$ -derivation, if for every $a \in A$ we have $\delta_x(a) = T(a)x - xS(a)$.

The Banach algebra A is said to be a $T-S$ -amenable, when for every Banach A -bimodule X , every $T-S$ -derivations from A into X^* is inner $T-S$ -derivations. The definition of weakly $T-S$ -amenable is similar.

Definition 2-2. Assume that A is a Banach algebra and $T : A \rightarrow A$ is a continuous linear mapping such that the mapping $b'' \rightarrow a''T''(b'') : A^{**} \rightarrow A^{**}$ is *weak*-to-weak* continuous where $a'' \in A^{**}$. Then we say that $a'' \in A^{**}$ has $T-w^*w$ property. We say that $B \subseteq A^{**}$ has $T-w^*w$ property, if every $b \in B$ has $T-w^*w$ property.

Let A be a Banach algebra and A^{**} has $I-w^*w$ property whenever $I : A \rightarrow A$ is the identity mapping. Then, obviously that A is Arens regular. There are some non-reflexive Banach algebras which the second dual of them have $T-w^*w$ property. If A is Arens regular, then, in general, A^{**} has not $I-w^*w$ property. In the following we give some examples from Banach algebras that the second dual of them have $T-w^*w$ property or no.

- (1) Let A be a non-reflexive Banach space and suppose that $\langle f, x \rangle = 1$ for some $f \in A^*$ and $x \in A$. We define the product on A as $ab = \langle f, a \rangle b$ for all $a, b \in A$. It is clear that A is a Banach algebra with this product, then A^{**} has $I-w^*w$ property whenever $I : A \rightarrow A$ is the identity mapping.
- (2) Every reflexive Banach algebra has $T-w^*w$ property.
- (3) Consider the algebra $c_0 = (c_0, \cdot)$ is the collection of all sequences of scalars that convergence to 0, with the some vector space operations and norm as ℓ_∞ . Then $c_0^{**} = \ell_\infty$ has $I-w^*w$ property whenever $I : c_0 \rightarrow c_0$ is the identity mapping.
- (4) $L^1(G)^{**}$ and $M(G)^{**}$ have not $I-w^*w$ property whenever G is locally compact group, but when G is finite, $L^1(G)^{**}$ and $M(G)^{**}$ have $I-w^*w$ property.

Theorem 2-3. Assume that A is a Banach algebra and $A^{(n+2)}$ has $T-w^*w$ property. If $D : A^{(n)} \rightarrow A^{(n+1)}$ is a $T-S$ -derivation, then $D'' : A^{(n+2)} \rightarrow A^{(n+3)}$ is a $T''-S''$ -derivation.

Proof. Let $a^{(n+2)}, b^{(n+2)} \in A^{(n+2)}$ and let $(a_\alpha^{(n)})_\alpha, (b_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n+2)}$ and $b_\beta^{(n)} \xrightarrow{w^*} b^{(n+2)}$. Due to $A^{(n+2)}$ has $T-w^*w$ property, we have $c^{(n+2)}T(a_\alpha^{(n)}) \xrightarrow{w} c^{(n+2)}T''(a^{(n+2)})$ for all $c^{(n+2)} \in A^{(n+2)}$. Using the *weak*-to-weak** continuity of D'' , we obtain

$$\begin{aligned} \lim_\alpha \lim_\beta \langle T(a_\alpha^{(n)})D(b_\beta^{(n)}), c^{(n+2)} \rangle &= \lim_\alpha \lim_\beta \langle D(b_\beta^{(n)}), c^{(n+2)}T(a_\alpha^{(n)}) \rangle \\ &= \lim_\alpha \langle D''(b^{(n+2)}), c^{(n+2)}T(a^{(n+2)}) \rangle = \langle D''(b^{(n+2)}), c^{(n+2)}T''(a^{(n+2)}) \rangle \\ &= \langle T''(a^{(n+2)})D''(b^{(n+2)}), c^{(n+2)} \rangle. \end{aligned}$$

Moreover, it is also clear that for every $c^{(n+2)} \in A^{(n+2)}$, we have

$$\lim_\alpha \lim_\beta \langle D(a_\alpha^{(n)})S(b_\beta^{(n)}), c^{(n+2)} \rangle = \langle D''(a^{(n+2)})S''(b^{(n+2)}), c^{(n+2)} \rangle.$$

Notice that in latest equalities, we didn't need $S-w^*w$ property for $A^{(n+2)}$. In the following, we take limit on the *weak** topologies. Thus we have

$$\begin{aligned} D''(a^{(n+2)}b^{(n+2)}) &= \lim_\alpha \lim_\beta D(a_\alpha^{(n)}b_\beta^{(n)}) = \lim_\alpha \lim_\beta T(a_\alpha^{(n)})D(b_\beta^{(n)}) + \\ &= \lim_\alpha \lim_\beta D(a_\alpha^{(n)})S(b_\beta^{(n)}) = T''(a^{(n+2)})D''(b^{(n+2)}) + D''(a^{(n+2)})S''(b^{(n+2)}). \end{aligned}$$

□

Theorem 2-4. Assume that A is a Banach algebra and $A^{(n+2)}$ has $T-w^*w$ property. If $A^{(n+2)}$ is weakly $T''-S''$ -amenable, then $A^{(n)}$ is weakly $T-S$ -amenable.

Proof. Let $D : A^{(n)} \rightarrow A^{(n+1)}$ is a $T-S$ -derivation, then by Theorem 2-3, $D'' : A^{(n+2)} \rightarrow A^{(n+3)}$ is a $T''-S''$ -derivation. Since $A^{(n+2)}$ is weakly $T''-S''$ -amenable, $D'' : A^{(n+2)} \rightarrow A^{(n+3)}$ is an inner $T''-S''$ -derivation. It follows that for every $a^{(n+2)} \in A^{(n+2)}$, we have

$$D''(a^{(n+2)}) = T''(a^{(n+2)})a^{(n+3)} - a^{(n+3)}S''(a^{(n+2)}).$$

for some $a^{(n+3)} \in A^{(n+3)}$. Take $a^{(n+1)} = a^{(n+3)}|_{A^{(n+1)}}$. Then for every $a^{(n)} \in A^{(n)}$, we have

$$D(a^{(n)}) = T(a^{(n)})a^{(n+1)} - a^{(n+1)}S(a^{(n)}).$$

It follows that D is inner $T-S$ -derivation, and so proof is hold.

□

Corollary 2-5. Let A be a Banach algebra and $I : A \rightarrow A$ be identity mapping. If A^{**} has $I-w^*w$ property and A^{**} is weakly amenable, then A is weakly amenable.

Corollary 2-6. Let A be a Banach algebra. If $A^{***}A^{**} \subseteq A^*$ and A^{**} is weakly amenable, then A is weakly amenable.

Proof. We show that A^{**} has $I - w^*w$ property where $I : A \rightarrow A$ is identity mapping. Suppose that $a'', b'' \in A^{**}$ and $b''_\alpha \xrightarrow{w^*} b''$. Let $c''' \in A^{***}$. Since $c'''a'' \in A^*$, we have

$$\langle c''', a''b''_\alpha \rangle = \langle c'''a'', b''_\alpha \rangle = \langle b''_\alpha, c'''a'' \rangle \rightarrow \langle b'', c'''a'' \rangle = \langle c''', a''b'' \rangle.$$

We conclude that $a''b''_\alpha \xrightarrow{w} a''b''$. So A^{**} has $I - w^*w$ property. By using Corollary 2-5, A is weakly amenable. \square

Example 2-7. c_0 is weakly amenable.

Proof. Since $\ell^\infty = c_0^{**}$ is weakly amenable and ℓ^∞ has $I - w^*w$ property by Corollary 2-5, proof is hold. \square

Theorem 2-8. Suppose that A is a Banach algebra and B is a closed subalgebra of $A^{(n+2)}$ that is consisting of $A^{(n)}$ where $n \in \mathbb{N} \cup \{0\}$. If B has $T - w^*w$ property and is weakly $T'' - S''$ -amenable, then $A^{(n)}$ is weakly $T - S$ -amenable.

Proof. Suppose that $D : A^{(n)} \rightarrow A^{(n+1)}$ is a $T - S$ -derivation and $p : A^{(n+3)} \rightarrow B'$ is the restriction map, defined by $P(a^{(n+3)}) = a^{(n+3)}|_{B'}$ for every $a^{(n+3)} \in A^{(n+3)}$. Since B has $T - w^*w$ property, $\bar{D} = PoD''|_B : B \rightarrow B'$ is a $T'' - S''$ -derivation. Since B is weakly $T'' - S''$ -amenable, there is $b' \in B'$ such that $\bar{D} = \delta_{b'}$. We take $a^{(n+1)} = b'|_{A^{(n+1)}}$, then $D = \bar{D}$ on $A^{(n+1)}$. Consequently, we have $D = \delta_{a^{(n+1)}}$. \square

Theorem 2-9. Let X be a Banach A -bimodule and let $T, S : A^{(n)} \rightarrow A^{(n)}$ be continuous linear mappings. Let the mapping $a^{(n+2)} \rightarrow x^{(n+2)}T''(a^{(n+2)})$ be $weak^*$ -to- $weak$ continuous for all $x^{(n+2)} \in X^{(n+2)}$. If $D : A^{(n)} \rightarrow X^{(n+1)}$ is a $T - S$ -derivation, then $D'' : A^{(n+2)} \rightarrow X^{(n+3)}$ is a $T'' - S''$ -derivation.

Proof. Let $a^{(n+2)}, b^{(n+2)} \in A^{(n+2)}$ and let $(a_\alpha^{(n)})_\alpha, (b_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n+2)}$ and $b_\beta^{(n)} \xrightarrow{w^*} b^{(n+2)}$. Then for all $x^{(n+2)} \in X^{(n+2)}$, we have $x^{(n+2)}T(a_\alpha^{(n)}) \xrightarrow{w} x^{(n+2)}a^{(n+2)}$. Consequently, we have

$$\begin{aligned} \lim_\alpha \lim_\beta \langle T(a_\alpha^{(n)})D(b_\beta^{(n)}), x^{(n+2)} \rangle &= \lim_\alpha \lim_\beta \langle D(b_\beta^{(n)}), x^{(n+2)}T(a_\alpha^{(n)}) \rangle \\ &= \lim_\alpha \langle D''(b^{(n+2)}), x^{(n+2)}T(a_\alpha^{(n)}) \rangle = \langle D''(b^{(n+2)}), x^{(n+2)}T(a^{(n+2)}) \rangle \\ &= \langle T(a^{(n+2)})D''(b^{(n+2)}), x^{(n+2)} \rangle. \end{aligned}$$

For every $x^{(n+2)} \in X^{(n+2)}$, we have also the following equalities

$$\begin{aligned} \lim_\alpha \lim_\beta \langle D(a_\alpha^{(n)})S(b_\beta^{(n)}), x^{(n+2)} \rangle &= \lim_\alpha \lim_\beta \langle D(a_\alpha^{(n)}), S(b_\beta^{(n)})x^{(n+2)} \rangle \\ &= \lim_\alpha \langle D(a_\alpha^{(n)}), S(b^{(n+2)})x^{(n+2)} \rangle = \langle D''(a^{(n+2)}), S(b^{(n+2)})x^{(n+2)} \rangle \\ &= \langle D''(a^{(n+2)})S(b^{(n+2)}), x^{(n+2)} \rangle. \end{aligned}$$

In the following, we take limit on the $weak^*$ topologies. Using the $weak^*$ -to- $weak^*$ continuity of D'' , we obtain

$$\begin{aligned} D''(a^{(n+2)}b^{(n+2)}) &= \lim_\alpha \lim_\beta D(a_\alpha^{(n)}b_\beta^{(n)}) = \lim_\alpha \lim_\beta T(a_\alpha^{(n)})D(b_\beta^{(n)}) + \\ &\quad \lim_\alpha \lim_\beta D(a_\alpha^{(n)})S(b_\beta^{(n)}) = T''(a^{(n+2)})D''(b^{(n+2)}) + D''(a^{(n+2)})S''(b^{(n+2)}). \end{aligned}$$

Thus $D'' : A^{(n+2)} \rightarrow X^{(n+3)}$ is a $T'' - S''$ -derivation. □

Corollary 2-10. Let X be a Banach A -bimodule and the mapping $a'' \rightarrow x''a''$ be $weak^* - to - weak$ continuous for all $x'' \in X^{**}$. Then, if $H^1(A^{**}, X^{***}) = 0$, it follows that $H^1(A, X^*) = 0$.

Corollary 2-11. Let X be a Banach A -bimodule and the mapping $a'' \rightarrow x''a''$ be $weak^* - to - weak$ continuous for all $x'' \in X^{**}$. If $D : A \rightarrow X^*$ is a derivation, then $D''(A^{**})X^{**} \subseteq A^*$.

Proof. By using Theorem 2-9 and [14, Corollary 4-3], proof is hold. □

Theorem 2-12. Let X be a Banach A -bimodule and $D : A \rightarrow X^*$ be a surjective derivation. Suppose that $D'' : A^{**} \rightarrow X^{***}$ is also a derivation. Then the mapping $a'' \rightarrow x''a''$ is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.

Proof. Let $a'' \in A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. We show that $x''a''_\alpha \xrightarrow{w} x''a''$ for all $x'' \in X^{**}$. Suppose that $x''' \in X^{***}$. Since $D''(A^{**}) = X^{***}$, by using [14, Corollary 4-3], we conclude that $X^{***}X^{**} = D''(A^{**})X^{**} \subseteq A^*$. Then $x'''x'' \in A^*$, and so we have the following equality

$$\langle x''', x''a''_\alpha \rangle = \langle x'''x'', a''_\alpha \rangle = \langle a''_\alpha, x'''x'' \rangle \rightarrow \langle a'', x'''x'' \rangle = \langle x''', x''a'' \rangle.$$

□

Corollary 2-13. Suppose that X is a Banach A -bimodule and A is Arens regular. Assume that $D : A \rightarrow X^*$ is a surjective derivation. Then $D'' : A^{**} \rightarrow X^{***}$ is a derivation if and only if the mapping $a'' \rightarrow x''a''$ from A^{**} into X^{**} is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.

Proof. By using Corollary 2-11, Theorem 2-12 and [14, Corollary 4-3], proof is hold. □

In the proceeding Corollary, if we omit the Arens regularity of A , then we have also the following conclusion.

Assume that $D : A \rightarrow X^*$ is a surjective derivation. Then, $D''(A^{**})X^{**} \subseteq A^*$ if and only if the mapping $a'' \rightarrow x''a''$ is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.

Corollary 2-14. Let A be a Banach algebra. Then we have the following results:

- (1) Assume that A is Arens regular and $D : A \rightarrow A^*$ is a surjective derivation. Then $D'' : A^{**} \rightarrow A^{***}$ is a derivation if and only if A has $I - w^*w$ property whenever $I : A \rightarrow A$ is the identity mapping.
- (2) Assume that $D : A \rightarrow A^*$ is a surjective derivation. Then, A has $I - w^*w$ property if and only if $D''(A^{**})A^{**} \subseteq A^*$. So it is clear that if $D : A \rightarrow A^*$ is a surjective derivation and $D''(A^{**})A^{**} \subseteq A^*$, then A is Arens regular.

Problem. Let S be a semigroup. Dose $C(S)^{**}$, $L^1(S)^{**}$ and $M(S)^{**}$ have $I - w^*w$ property? whenever I is the identity mapping.

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